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The twistor theory of Whitham hierarchy

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Abstract

We have generalized the approach of M. Dunajski, L. Mason and P. Tod [Einstein–Weyl geometry, the dKP equation and twistor theory, J. Geom. Phys. 37 (2001) 63–93] and established a 1–1 correspondence between a solution of the Universal Whitham hierarchy [I.M. Krichever, The τ-function of the universal Whitham hierarchy, matrix models and topological field theories, Comm. Pure Appl. Math. 47 (1994) 437–475] and a twistor space. The twistor space consists of a complex surface and a family of complex curves together with a meromorphic 2-form. The solution of the Universal Whitham hierarchy is given by deforming the curve in the surface. By treating the family of algebraic curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a twistor space, we were able to express the deformations of the isomonodromic spectral curve in terms of the deformations generated by the Universal Whitham hierarchy.

MSC: 32L25

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1. Introduction

The Whitham hierarchy originates from the study of the 'dispersionless limit' of integrable systems. The typical set-up involves an introduction of a small parameter ϵ and a suitable rescaling of the time variables. For example, if one rescales the times x, t to $\tilde{x} = \epsilon x$, $\tilde{t} = \epsilon t$ in the KdV equation

$$\partial_t u = 6u\partial_x u - \partial_x^3 u$$

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one obtains the KdV equation with small dispersion

$$\tilde{\partial}_t u(\tilde{x}, \tilde{t}, \epsilon) = 6u\tilde{\partial}_x u(\tilde{x}, \tilde{t}, \epsilon) - \epsilon^2 \tilde{\partial}_x^3 u(\tilde{x}, \tilde{t}, \epsilon) \tag{1}$$

where $\tilde{\partial}_x$ and $\tilde{\partial}_t$ denote the derivatives with respect to the tilded variables.

The problem is to study the behavior of the solution $u(\tilde{x}, \tilde{t}, \epsilon)$ as $\epsilon \to 0$. In general, the solution contains parts that oscillates rapidly as $\epsilon \to 0$ and the limit does not exist.

However, the weak limit of $u(\tilde{x}, \tilde{t}, \epsilon)$ can be studied. This is like 'averaging out' the oscillatory part and studying the limit, and such an averaging process is called Whitham averaging [6,7,14, 11,12,21,22,24–26,33].

There is an interesting connection between the small dispersion KdV and deformations of Riemann surfaces [12].

A finite gap solution of the KdV equation can be expressed in terms of theta functions of a genus g algebraic curve (the spectral curve). The spectral curve is fixed under the evolution of the finite gap solution [5,19,20]. In many applications, a solution $u(x, t)$ of the KdV equation can be approximated by a finite gap solution locally. However, the approximation becomes bad as (x, t) grows large and the spectral curve that classifies the finite gap solution varies slowly with respect to (x, t). To study the variations of the spectral curve, one could think of the spectral curve as depending on the 'slow times' $(\tilde{x} = \epsilon x, \tilde{t} = \epsilon t)$ and study the dependence of u on $(\tilde{x}, \tilde{t})$ in (1) as $\epsilon \to 0$.

In this case, the weak limit of $u(\tilde{x}, \tilde{t}, \epsilon)$ can be described by $2g + 1$ functions that satisfy the g-phase Whitham equations [33]

$$\tilde{\partial}_t u_i - v_{g,i}(u_1, \ldots, u_{2g+1})\tilde{\partial}_x u_i = 0, \quad 1 = 1, 2, \ldots, 2g + 1 \tag{2}$$

where $v_{g,i}(u_1, \ldots, u_{2g+1})$ depends on u_i on complete hyperelliptic integrals of genus g. These equations can in fact be written in a more compact form

$$\tilde{\partial}_t \mathrm{d}p = \tilde{\partial}_x \mathrm{d}q \tag{3}$$

where $\mathrm{d}p$ and $\mathrm{d}q$ are certain meromorphic 1-forms that are normalized on the spectral curve with respect to a choice of a canonical basis of cycles. To be precise, the spectral curve in this case is a hyperelliptic curve

$$w^2 = R(z) = \prod_{i=1}^{2g+1} (z - r_i)$$

and $\mathrm{d}p$, $\mathrm{d}q$ are 1-forms of the form

$$\mathrm{d}p = \frac{\sum_{i=1}^{g} c_i z^i}{\sqrt{R(z)}}$$
$$\mathrm{d}q = \frac{\sum_{i=1}^{g+1} d_i z^i}{\sqrt{R(z)}}$$
$$\int_{r_{2i}}^{r_{2i-1}} \mathrm{d}p = \int_{r_{2i}}^{r_{2i-1}} \mathrm{d}q = 0.$$

To obtain (2) from (3), we expand $\mathrm{d}p$ and $\mathrm{d}q$ in terms of z and the coefficients would satisfy (2) [11,12,6,21].

Krichever has studied equations of the type (3) on more general algebraic curves that cover the dispersionless limit of other integrable systems and introduced the notion of the Universal Whitham hierarchy [23]. The Universal Whitham hierarchies, like (3), are expressed in terms of meromorphic and also holomorphic 1-forms on Riemann surfaces. It has many interesting applications to 2-D topological quantum field theory, Frobenius manifolds and string theory [1–4,22,23,32]. Throughout this paper, we will use the term Universal Whitham hierarchy to denote the system of differential equations defined in Definition 3, while the term Whitham hierarchy will be reserved for the system of PDE that comes from the Whitham averaging procedure.

In the case where the curve in the question is of genus 0, Dunajski, Mason and Tod have studied the dispersionless KP (dKP) equation from a point of view of twistor theory, in which a solutions of the dKP equation is described by a family of rational curves in a complex surface and a meromorphic 2-form on the surface. This reveals interesting relations between the dKP equation and the Einstein–Weyl metric [8–10]. In [13], special solutions of the dKP equation was constructed by using twistor methods.

In this paper, we have generalized the construction of [10] to the case where the curves are of arbitrary genus. We have established a one–one correspondence between a solution of the Universal Whitham hierarchy and a twistor space, which is a complex surface with a family of genus g curves in it. To be precise, we have the following

Definition 1. A twistor space $\mathcal{T}$ of the truncated Universal Whitham hierarchy consists of:

1. A 2-dimensional complex manifold $\mathcal{T}$ with a meromorphic 2-form Π which is singular on a divisor D,
2. A family of genus-g embedded curves $\{\Sigma_{g,t}\}$ and canonical basis of cycles on each curve,
3. A covering V_β of a neighborhood U of the singular divisor D, and local coordinates k_β^{-1} on V_β such that k_β^{-1} has zeros of order 1 at D. A fix point P_1 on each curve Σ_t such that $P_1 \in D$

and

Theorem 1. *There is a one–one correspondence between a solution of the Universal Whitham hierarchy and a twistor space $\mathcal{T}$ defined above.*

The idea is that as the curve moves around in the twistor space, the values of the 2-form Π evaluated on different curves would give the values of the solution at different times.

We have also studied the following problem which relates the isomonodromic spectral curve to the Universal Whitham hierarchy. Consider the isomonodromic deformations of the system of linear ODEs

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = A(z)Y$$

where A is an $n \times n$ matrix such that

$$A(z) = G^{\mathrm{T}}(\Delta - z)^{-1}F + C$$

where G, F are $n \times r$ matrices constant in z, $\Delta = \mathrm{diag}(\alpha_1, \ldots, \alpha_r)$ and $C = \mathrm{diag}(c_1, \ldots, c_n)$.

Define the *dual isomonodromic spectral curve* [15,27] of this isomonodromic system by

$$\det\left[\mathbb{M} - \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}\right] = 0.$$

These spectral curves are known to vary as the matrix $A(z)$ deforms isomonodromically.

The family of spectral curves is a family of algebraic curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ defined by polynomials $P(z, w)$ that has degree at most n in w and r in z. Since the family of curves defined by all polynomials $P(z, w)$ with degree at most n in w and r in z forms a twistor space for the Universal Whitham hierarchy, we can compare the changes of the spectral curve induced by isomonodromic deformations and the changes induced by the Whitham hierarchy. This problem was posted by Takesaki in [28,29] in a slightly different form. We were able to derive formulas that express the isomonodromic flows in terms of the Universal Whitham flows in Proposition 8.

This paper is organized as follows. In Section 2 we will give a brief introduction to the Universal Whitham hierarchy and establish some notations. In Section 3 we will construct the twistor space for the Universal Whitham hierarchy and in Section 4 we will give some examples and study the isomonodromic spectral curves.

2. The Universal Whitham hierarchy

In this section we will give a brief introduction to the Universal Whitham hierarchy in [23].

Let M_{g,N,n_α} be the modulus space of genus g curves $\Gamma_{g,q}$ with N punctures $P_\alpha(q)$, fixed n_α-jets of local coordinates $k_\alpha^{-1}(q)$ in neighborhoods of $P_\alpha(q)$ and canonical basis of cycles $a_i(q)$, $b_i(q)$. The marked points, local coordinates and basis change from curve to curve, but the genus g of the curves, the total number of marked points N and the numbers n_α do not vary between curves. Let $\sum_\alpha (n_\alpha + 1) = H$ then the dimension of the modulus space M_{g,N,n_α} is $3g - 2 + H$.

That is, we have

$$M_{g,N} = \{\Gamma_{g,q}, P_\alpha(q), k_\alpha^{-1}(q), a_i(q), b_i(q) \in H_1(\Gamma_{g,q}, \mathbb{Z})\}.$$

The tautological bundle $\hat{M}_{g,N,n_\alpha} \to M_{g,N,n_\alpha}$ is the fiber bundle over M_{g,N,n_α} with the fiber at each point q of M_{g,N,n_α} being the curve $\Gamma_{g,q}$. The curves $\Gamma_{g,q}$ at different points are not holomorphic to each other in general.

We consider each fiber $\Gamma_{g,q}$ as a genus g Riemann surface with canonical basis $\{a_i(q), b_i(q)\}$ of cycles, marked points $P_\alpha(q)$ and local coordinates $k_\alpha^{-1}(q)$ near $P_\alpha(q)$. We now define various meromorphic 1-forms $\mathrm{d}\Omega_{\mu,\nu}(q)$ on these Riemann surfaces according to the following

Definition 2. The meromorphic 1-forms $\mathrm{d}\Omega_A$ where $A = (\mu, \nu)$ a double index are defined by

1. $\mathrm{d}\Omega_{\alpha,i}(q)$ are meromorphic 1-forms of the second kind which are holomorphic outside $P_\alpha(q)$ and have the form
$$\mathrm{d}\Omega_{\alpha,i}(q) = \mathrm{d}(k_\alpha^i(q) + O(k_\alpha^{-1}(q))) \tag{4}$$
near $P_\alpha(q)$, where $i = 1, \ldots, n_\alpha$.
2. $\mathrm{d}\Omega_{\alpha,0}(q)$, $\alpha \neq 1$ are meromorphic 1-forms of the third kind with residues 1 and -1 at $P_1(q)$ and $P_\alpha(q)$ respectively, and they look like
$$\begin{aligned}\mathrm{d}\Omega_{\alpha,0} = {} & \mathrm{d}k_\alpha(q)(k_\alpha^{-1}(q) + O(k_\alpha^{-1}(q))) \\ & - \mathrm{d}k_1(q)(k_1^{-1}(q) + O(k_1^{-1}(q)))\end{aligned} \tag{5}$$
near $P_\alpha(q)$ and $P_1(q)$.
3. $\mathrm{d}\Omega_{h,k}(q)$, ($k = 1, \ldots, g$ and h fixed) are the normalized holomorphic 1-forms
$$\oint_{a_i(q)} \mathrm{d}\Omega_{h,k}(q) = \delta_{i,k}, \quad i, k = 1, \ldots, g. \tag{6}$$

4. The differentials in 1 and 2 are uniquely determined by the normalization conditions

$$\oint_{a_i(q)} \mathrm{d}\Omega_A(q) = 0 \tag{7}$$

where the index $A = (\alpha, i)$.

We now fix an index $(\alpha, i) = (1, 1)$ and denote $\mathrm{d}\Omega_{1,1}(q)$ by $\mathrm{d}p(q)$. Let the multi-valued function $p(q)$ be defined by the abelian integral of $\mathrm{d}p(q)$

$$p(P, q) = \int^P \mathrm{d}p(P, q) \tag{8}$$

where $P \in \Gamma_{g,q}$. We will now suppress the dependence of q and bear in mind that everything depends on the point q of the modulus space. We can expand the local coordinates k_α in terms of the function p near each marked point P_α.

$$k_\alpha = \sum_{s=-1}^{\infty} \nu_{\alpha,s}(p - p_\alpha)^s, \quad \alpha \neq 1 \tag{9}$$

where $p_\alpha = p(P_\alpha)$, and for $\alpha = 1$

$$k_1 = p + \sum_{s=1}^{\infty} \nu_{1,s} p^s. \tag{10}$$

We can now use the parameters

$$\begin{aligned} &\{p_\alpha = p(P_\alpha), \nu_{\alpha,s}, \alpha = 1, \ldots, N, s = -1, \ldots, n_\alpha\} \\ &\sigma_s = p(\Sigma_s), \quad \mathrm{d}p(\Sigma_s) = 0, \quad s = 1, \ldots, 2g-2 \\ &U_i^p = \oint_{b_i} \mathrm{d}p, \quad i = 1, \ldots, g \end{aligned} \tag{11}$$

as coordinates on the modulus space M_{g,N,n_α}. Note that we only need the coefficients $\nu_{\alpha,s}$ up to $s = n_\alpha$ as we are considering n_α-jets, which means that terms of order k_α^i, $i > n_\alpha$ are all equivalent and do not make any difference.

Suppose we have some *unknown* functions $t_{\mu,\nu}$ on the modulus space M_{g,N,n_α} which are in 1–1 correspondence with the 1-forms $\mathrm{d}\Omega_{\mu,\nu}$. That is, the $t_{\mu,\nu}$ are unknown functions of the coordinates (11). Also, suppose these functions $t_{\mu,\nu}$ form a coordinate system on a subspace M' of M_{g,N,n_α}. Let $\Omega_{\mu,\nu}$ be the abelian integrals of $\mathrm{d}\Omega_{\mu,\nu}$ as in (8). Let $\hat{M}'$ be the tautological bundle over M'. We now construct a 2-form Π on $\hat{M}'$, by using the $t_{\mu,\nu}$ and $\Omega_{\mu,\nu}$ as follows

$$\Pi = \sum_A \delta\Omega_A \wedge \mathrm{d}t_A \tag{12}$$

where for simplicity, we write $(\mu, \nu) = A$ and

$$\delta\Omega_A = \partial_p \Omega_A \mathrm{d}p + \sum_B \partial_B \Omega_A \mathrm{d}t_B.$$

When performing the derivative, we treat t_A and p as independent variables on $\hat{M}'$.

Krichever's construction of the Universal Whitham hierarchy is then equivalent to the following. One wants to ask: how should the t_A depend on (11) if we want Π to be simple?

$$\Pi \wedge \Pi = 0. \tag{13}$$

The coefficients of $\mathrm{d}p \wedge \mathrm{d}t_A \wedge \mathrm{d}t_B \wedge \mathrm{d}t_C$ of $\Pi \wedge \Pi$ give the following set of partial differential equations

$$\sum \epsilon_{ABC} \partial_A \Omega_B \partial_p \Omega_C = 0$$

where the summation is taken over all the permutations of A, B and C and ϵ_{ABC} is the sign of the permutation. By taking $C = (1, 1)$, and denote $t_{1,1}$ by x, we get the following

$$\partial_A \Omega_B - \partial_B \Omega_A + \{\Omega_A, \Omega_B\} = 0 \tag{14}$$

where $\{f, g\}$ is the Poisson bracket with respect to the symplectic form $\mathrm{d}x \wedge \mathrm{d}p$.

This set of partial differential equations is called the truncated Universal Whitham hierarchy $W(n_\alpha)$.

By expanding both sides of these equations in terms of p, we obtain partial differential equations of the coordinates (11) with respect to the t_A. By solving these equations, we obtain the coordinates (11) as functions of t_A and also define M'. Of course, the t_A do not form a complete set of coordinates on the modulus space M_{g,N,n_α}, so the truncated Universal Whitham equations in fact give the coordinates of the modulus space in terms of the t_A on the subspace M' where the t_A becomes a complete set of coordinates.

The Eqs. (14) should be understood as a system of PDE relating the t_A and the coordinates (11), which enter the equations as the coefficients of the multi-valued 'functions' Ω_A expanding in terms of the multi-valued 'coordinate' p on the curve $\Gamma_{g,q}$. Namely, we choose branches of the functions Ω_A and p such that the expansions give the coordinates (11). The Eqs. (14) should only be understood locally. Having this interpretation in mind, we will now take a closer look at the consequence on (14).

Proposition 2. *The truncated Universal Whitham hierarchy* (14) *gives*

$$\begin{aligned} &\partial_A U_i^p = \partial_x U_i^A, \quad U_i^A = \oint_{b_i} \mathrm{d}\Omega_A, \quad i = 1, \ldots, g \\ &\partial_A \sigma_s = \partial_x \Omega_A (\Sigma_s), \quad s = 1, \ldots, 2g-2 \end{aligned} \tag{15}$$

after expanding the terms near σ_s and taking the b-periods, where the 1-forms are defined in Definition 2.

Proof. Consider the b-periods of Eqs. (14). First note that since $p \mapsto p + U_i^p$ after going around the cycle b_i, the vector fields ∂_A becomes $\partial_A \mapsto \partial_A + \partial_A U_i^p \partial_p$ after going around a b_i-cycle. The b_i-period of (14) is therefore

$$(\partial_A U_i^p - \partial_x U_i^A)\partial_p \Omega_B - (\partial_B U_i^p - \partial_x U_i^B)\partial_p \Omega_A + \partial_A U_i^B - \partial_B U_i^A. \tag{16}$$

By multiplying both sides by $\mathrm{d}p$, we can view these as equations of 1-forms. Since the 1-forms $\mathrm{d}\Omega_{\mu,\nu}$ are linearly independent, the coefficient of $\mathrm{d}\Omega_B$ must vanish, which gives the first set of equations in (22).

The second set of equations is obtained by looking at the behavior of (14) near the branch points σ_s. Near the branch points σ_s, we have to use $(p - \sigma_s)^{\frac{1}{2}}$ as a local coordinate. We expand the functions Ω_A near σ_s as follows

$$\Omega_A = \Omega_A(\Sigma_s) + \beta_{A,1}(T)(p-\sigma_s)^{\frac{1}{2}} + \beta_{A,2}(T)(p-\sigma_s) + \cdots. \tag{17}$$

We now expand (14) near the point σ_s. The term $\partial_A \Omega_B$ is of the form

$$\partial_A \Omega_B = -\partial_A \sigma_s \partial_p \Omega_B + O(1) \tag{18}$$

and the term $\partial_x \Omega_A \partial_p \Omega_B$ is of the form

$$\begin{aligned}\partial_x \Omega_A \partial_p \Omega_B = {} & \partial_x \Omega_A(\sigma_s)\partial_p \Omega_B \\ & + \beta_{A,1}(T)\beta_{B,1}(T)\left(-\frac{1}{4}\right)\partial_x \sigma_s (p-\sigma_s)^{-1} \\ & - \left(\frac{1}{2}\beta_{A,1}(T)\beta_{B,2}(T)\partial_x \sigma_s \right. \\ & \left. + \frac{1}{2}\beta_{A,2}(T)\beta_{B,1}(T)\partial_x \sigma_s\right)(p-\sigma_s)^{-\frac{1}{2}} \\ & + O(1)\end{aligned} \tag{19}$$

since the second and the third terms in the right hand side is symmetric in A and B, the expansion of (14) near σ_s is then

$$(-\partial_A \sigma_s + \partial_x \Omega_A(\Sigma_s))\partial_p \Omega_B + (-\partial_B \sigma_s + \partial_x \Omega_B(\Sigma_s))\partial_p \Omega_A + O(1) \tag{20}$$

we now choose Ω_B such that $\mathrm{d}\Omega_B(\Sigma_s) \neq 0$. For example, the holomorphic 1-forms $\mathrm{d}\Omega_{h,k}$ only have $2g-2$ zeros while the number of points Σ_s is $2g$. Therefore there exists some $B=(h,k)$ such that $\mathrm{d}\Omega_B(\Sigma_s) \neq 0$. By a similar argument as before, we arrive at the second set of equations. □

We also want the truncated Universal Whitham hierarchy $W(n_\alpha)$ to be embedded in an infinite hierarchy with $n_\alpha \to \infty$. This would give us the following

Proposition 3. *As $n_\alpha \to \infty$ for all α, the truncated Universal Whitham hierarchy (14) gives*

$$\partial_A k_\alpha - \{k_\alpha, \Omega_A\} \to 0 \tag{21}$$

for the 1-forms Ω_A defined in Definition 2.

Proof. The proof can be found in [23]. Let $B=(\alpha,j)$ and $A=(\beta,i)$ in (14). Let X_- be the holomorphic part of X at P_α. Then we have

$$\partial_A k_\alpha^j - \{k_\alpha^j, \Omega_A\} = \partial_B \Omega_A - \{\Omega_A, (\Omega_B)_-\}.$$

Since $A=(\beta,i)$ and $(\Omega_B)_-$ is holomorphic at P_α, the worst pole that the right hand side could have at P_α is of order i. Therefore

$$\partial_A k_\alpha^j - \{k_\alpha^j, \Omega_A\} = O(k_\alpha^{i-j}).$$

By letting $j \to \infty$, the proposition is proved. □

We can now define the truncated Universal Whitham hierarchy as a system of PDE for the coordinates (11).

Definition 3 (*Truncated Universal Whitham Hierarchy*). The truncated Universal Whitham hierarchy $W(n_\alpha)$ is defined as the following system of PDEs

$$\begin{aligned}&\partial_A k_\alpha + \{\Omega_A, k_\alpha\} = 0\\&\partial_A U_i^p = \partial_x U_i^A, \quad U_i^A = \oint_{b_i} \mathrm{d}\Omega_A, \quad i = 1, \ldots, g\\&\partial_A \sigma_s = \partial_x \Omega_A(\Sigma_s), \quad s = 1, \ldots, 2g-2\end{aligned} \tag{22}$$

for the 1-forms defined in Definition 2.

To define these equations, we only use the coefficients of the Ω_A up to the p^{n_α} term. Since these coefficients are expressible in terms of (11), the PDE (22) defines the functional relation between the t_A and (11).

The set of differential equations (14) with indices $A, B = (\alpha, i)$ only (that is, excluding the indices (h, k) in 4) are called the Whitham hierarchy, they represent the dispersionless limits of integrable hierarchies.

For simplicity, we would, from now on, drop the word truncated and simply call $W(n_\alpha)$ the Universal Whitham hierarchy.

Remark. Although the vector fields ∂_A are multi-valued on $\hat{M}_{g,N,n_\alpha}$, they are, nevertheless, well-defined as normal vectors of the curve $\Gamma_{g,q} \subset \hat{M}_{g,N,n_\alpha}$. This turns out to be crucial in our construction of the twistor space.

2.1. Generating form of the Universal Whitham hierarchy

The Eqs. (14) can be considered as compatibility conditions of the system

$$\partial_A E_\beta = \{E_\beta, \Omega_A\} \tag{23}$$

where E_β is defined on some open set V_β on $\Gamma_{g,q}$. We can perform another change of variable and view E_β, t_A as independent variables. The set of Eqs. (14) then becomes

$$\partial_A \Omega_B(E_\beta, t) = \partial_B \Omega_A(E_\beta, t) \tag{24}$$

in these new coordinates. Eqs. (24) suggest the existence of a potential $S(E_\beta, t)$ such that

$$\partial_A S_\beta(E_\beta, t) = \Omega_A(E_\beta, t). \tag{25}$$

The 2-form Π is then

$$\Pi = \delta E_\beta \wedge \delta Q_\beta \tag{26}$$

where $Q_\beta = \partial_{E_\beta} S_\beta$.

The functions Q_β and E_β satisfy a ‘string equation’

$$\{Q_\beta, E_\beta\} = 1 \tag{27}$$

this implies that Q_β is also a solution of (23)

$$\partial_A Q_\beta = \{Q_\beta, \Omega_A\}. \tag{28}$$

3. The twistor space of the Universal Whitham hierarchy (genus-g curves)

We will now proceed to construct the twistor space of the Universal Whitham hierarchy. We will first construct a twistor space from the Universal Whitham hierarchy and study its properties. We will then extract these properties to define a twistor space and recover the Universal Whitham hierarchy from it. In constructing the twistor space (the direct construction), we will assume that we are given a solution to the Universal Whitham hierarchy and therefore we know the functional relation between t_A and the coordinates (11) and also the subspace M' of M_{g,N,n_α} where t_A forms a complete set of coordinates. We will denote the tautological bundle over M' by $\hat{M}'$. In particular, the 2-form Π is defined on $\hat{M}'$.

3.1. The symplectic reduction of M'

Following the spirit of [10], we will treat the 2-form Π as a presymplectic form on $\hat{M}'$ and take the symplectic reduction of $(\hat{M}', \Pi)$. This then gives us a 2-dimensional manifold which we will call the twistor space of the Universal Whitham hierarchy.

We first study the kernel of the 2-form Π. Due to the simplicity condition (13), the kernel is generically of codimension 2. We have the following

Proposition 4. *The kernel of the 2-form Π is spanned by the vector fields*

$$\partial_A - \partial_x \Omega_A \partial_p + \partial_p \Omega_A \partial_x. \tag{29}$$

Proof. This can be verified by direct calculation. We first contract ∂_A with Π, where we use (p, t_A) as our independent variables.

$$\partial_A \lrcorner \Pi = \sum_{B \neq (1,1)} (\partial_A \Omega_B - \partial_B \Omega_A) \mathrm{d}t_B - \partial_p \Omega_A \mathrm{d}p - \partial_x \Omega_A \mathrm{d}x.$$

We now apply (14) to the above equation and get

$$\partial_A \lrcorner \Pi = \sum_{B \neq (1,1)} (\partial_x \Omega_B \partial_p \Omega_A - \partial_x \Omega_A \partial_p \Omega_B) \mathrm{d}t_B - \partial_p \Omega_A \mathrm{d}p - \partial_x \Omega_A \mathrm{d}x. \tag{30}$$

We then consider the contraction between Π and ∂_p, ∂_x respectively.

$$\begin{aligned} \partial_p \lrcorner \Pi &= \mathrm{d}x + \sum_B \partial_p \Omega_B \mathrm{d}t_B \\ \partial_x \lrcorner \Pi &= -\mathrm{d}p + \sum_B \partial_x \Omega_B \mathrm{d}t_B. \end{aligned} \tag{31}$$

By comparing these with (30), we see at once that

$$(\partial_A - \partial_x \Omega_A \partial_p + \partial_p \Omega_A \partial_x) \lrcorner \Pi = 0.$$

This concludes the proof of the theorem. □

We can now consider the distribution Y spanned by the vector fields (29). Since these span the kernel of a closed 2-form, this distribution is integrable. Note that although the 2-form Π is degenerate and singular on some codimension-1 sets, this distribution is still well-defined on these sets. We would like to consider the leaf space of this distribution. That is, we would like to study the space $\hat{M}'/Y$. This will be our twistor space $\mathcal{T}$.

Remark. Note that the distribution Y is well-defined despite the fact that both ∂_A and Ω_A are not single valued. The vector field $\partial_A - \partial_x \Omega_A \partial_p + \partial_p \Omega_A \partial_x$ is defined up to an addition of

$$\partial_A U_i^A \partial_p - \partial_x U_i^p \partial_p - \partial_x U_i^p \partial_p \Omega_A \partial_p + \partial_p \Omega_A \partial_x U_i^A \partial_p$$

which vanishes due to the second set of equations in Definition 3. The distribution is therefore well-defined.

We can now study the properties of $\mathcal{T}$. The following gives us a more concrete picture of what $\mathcal{T}$ looks like.

Proposition 5. *The variables E_β, Q_β and k_α are functions on $\mathcal{T}$. In particular, if we use them as local coordinates on $\mathcal{T}$, the 2-form Π becomes $\Pi = \mathrm{d}E_\beta \wedge \mathrm{d}Q_\beta$ on $\mathcal{T}$.*

Proof. The proof is nothing more than unwinding the definition of $\{f, g\}$ in Eqs. (23), (28) and (22), which says that given a vector field $X \in Y$, $X(E) = X(Q) = X(k_\alpha) = 0$. $\square$

There is also a family of embedded curves in $\mathcal{T}$, which are projections of the curves $\Gamma_{g,q}$. These curves are labelled by the parameter t_A. In explicit terms, these embeddings are given by

$$Q_\beta = \partial_{E_\beta} S_\beta \tag{32}$$

as this equation is defined on the curves $\Gamma_{g,q}$ for $t_A = constant$.

The 2-form Π descends to the twistor space $\mathcal{T}$ to give a meromorphic section Π of the canonical bundle of $\mathcal{T}$, which is non-degenerate except on a codimension-1 subspace. Given a section T of the normal bundle N_q of a embedded curve $\Gamma_{g,q} \subset \mathcal{T}$, the map $\mu_T = T \lrcorner \Pi|_{\Gamma_{g,q}}$ gives a meromorphic 1-form on $\Gamma_{g,q}$ which can only have poles at the singular set of $\mathcal{S}$ of Π with order less than or equal to that of Π. This then defines an isomorphism between $H^0(\Gamma_{g,q}, N_q)$ and $H^0(\Gamma_{g,q}, K \otimes [D])$, where K is the canonical bundle of the $\Gamma_{g,q}$ and $[D]$ is the line bundle associated with the pole divisor of Π. Therefore we have the following

Proposition 6. *The normal bundle of an embedded curve $\Gamma_{g,q} \in \mathcal{T}$ is isomorphic to $K \otimes [D]$, where K is the canonical bundle of $\Gamma_{g,q}$ and $[D]$ is the pole divisor of Π.*

In fact, the 1-forms $\partial_A \lrcorner \Pi$ are nothing but the 1-forms $-\mathrm{d}\Omega_A$, since we have

$$\begin{aligned}\partial_A \lrcorner \Pi|_\Sigma &= (\partial_x \Omega_A \partial_p - \partial_p \Omega_A \partial_x) \lrcorner (\mathrm{d}E \wedge \mathrm{d}Q)|_\Sigma \\ &= \{(\partial_x Q \partial_p \Omega_A - \partial_p Q \partial_x \Omega_A)\partial_p E - (\partial_x E \partial_p \Omega_A - \partial_p E \partial_x \Omega_A)\partial_p Q\}\mathrm{d}p\end{aligned}$$

as $\mathrm{d}t_A|_\Sigma = 0$. We now apply the string equation (27) to obtain

$$\partial_A \lrcorner \Pi|_\Sigma = \partial_p \Omega_A \mathrm{d}p = -\mathrm{d}\Omega_A. \tag{33}$$

3.2. The twistor space of the Universal Whitham hierarchy

This concludes our study of the properties of $\mathcal{T}$. We will now define a twistor space independently and show that it gives a solution to the Universal Whitham hierarchy. We first define the twistor space of a Universal Whitham hierarchy.

Definition 4. A twistor space $\mathcal{T}$ of the truncated Universal Whitham hierarchy consists of:

1. A 2-dimensional complex manifold $\mathcal{T}$ with a meromorphic 2-form Π which is singular on a divisor D,

2. A family of genus-g embedded curves $\{\Sigma_{g,t}\}$ and canonical basis of cycles on each curve,
3. A covering V_β of a neighborhood U of the singular divisor D, and local coordinates k_β^{-1} on V_β such that k_β^{-1} has zeros of order 1 at D. A fix point P_1 on each curve Σ_t such that $P_1 \in D$.

The main result in this section is the following

Theorem 7. *There is a one–one correspondence between a solution of the Universal Whitham hierarchy and a twistor space $\mathcal{T}$ defined above.*

Proof. Given a solution to the Universal Whitham hierarchy, we can take the symplectic reduction of its modulus space as we did in the last section to get a twistor space $\mathcal{T}$. The local coordinates k_α near each marked point P_α are well-defined on the twistor space due to (21) and they give the local coordinates in 3 of Definition 4.

This gives the first half of the proof. To go the other way round, we want to recover the space $\hat{M}'$ and show that the pull-back $\tilde{\Pi}$ of the 2-form Π to $\hat{M}'$ has the correct form. The simplicity condition follows automatically as $\tilde{\Pi}$ is the pull-back of a 2-form on a 2-dimensional manifold.

The space M' is the set of curves $\{\Sigma_{g,t}\}$. By considering E and Q as functions of the t_A, we can pull back the 2-form Π to the space $\hat{M}_{g,N}$. This can be achieved by expanding E and Q near the divisor D in terms of the local coordinate

$$p = -\int \mathrm{d}\Omega_{1,1}$$

and considering the coefficients as functions of t.

Now choose representatives ∂_A of normal vectors such that

$$\partial_A \lrcorner \Pi = (\partial_x \Omega_A \partial_p - \partial_p \Omega_A \partial_x) \lrcorner \Pi \tag{34}$$

where $\partial_x \lrcorner \Pi = -\mathrm{d}p$. We see that

$$(\partial_x \Omega_A \partial_p - \partial_p \Omega_A \partial_x) \lrcorner \Pi|_\Sigma = \partial_p \Omega_A \mathrm{d}p$$

since $\mu_{\partial_x} = -\mathrm{d}p$. We see that t_A is a complete set of coordinates on M' because the ∂_A generate all the possible deformations of the curves and are independent.

We have

$$\tilde{\Pi} = \left(\sum_A \partial_A E \mathrm{d}t_A + \partial_p E \mathrm{d}p\right) \wedge \left(\sum_B \partial_B Q \mathrm{d}t_B + \partial_p Q \mathrm{d}p\right). \tag{35}$$

We now contract $\tilde{\Pi}$ with ∂_A and make use of (34). The $\mathrm{d}t_B$ component of $\partial_A \lrcorner \tilde{\Pi}$ then gives

$$\partial_x \Omega \partial_p E \partial_B Q - \partial_p Q \partial_x \Omega_A \partial_B E + \partial_p \Omega_A \partial_x E \partial_B Q - \partial_p \Omega_A \partial_x Q \partial_B E \tag{36}$$

while the $\mathrm{d}p$ component gives

$$\partial_p \Omega_A \partial_x E \partial_p Q - \partial_p \Omega_A \partial_x Q \partial_p E \tag{37}$$

since this is just $-\partial_p \Omega_A$, we obtain the string equation (27). Note that (34) can be written as

$$(\partial_A - \partial_x \Omega_A \partial_p + \partial_p \Omega_A \partial_x)(E)\mathrm{d}Q = (\partial_A - \partial_x \Omega_A \partial_p + \partial_p \Omega_A \partial_x)(Q)\mathrm{d}E$$

this gives (23) and (28), which implies the compatibility of the flows (14). By applying (23) and (28)–(36) and making use of the string equation (27), the expression (36) becomes

$$\partial_x \Omega_A \partial_p \Omega_B - \partial_p \Omega_A \partial_x \Omega_B = \partial_B \Omega_A - \partial_A \Omega_B \tag{38}$$

this proves the theorem follows. □

Remark. As in the case of the dKP equation [10], the Whitham hierarchy depends on the choice of coordinates, which is reflected in 3 of Definition 4.

4. Examples

We will now look at a few examples of twistor spaces of Universal Whitham hierarchies.

Example 1. The dKP equation [10].

The first example is the dKP equation, its twistor space was first constructed in [10] from the Einstein–Weyl metric. This is the case of a Whitham hierarchy with genus-0 curves. In this case, the Universal Whitham hierarchy and the Whitham hierarchy become the same.

The twistor space of this equation is given by a 2-dimensional complex manifold $\mathcal{T}$ such that the 2-form Π is singular at a connected set D with order 4. The Darboux coordinates E, Q are chosen so that E, Q are singular on D with orders 1 and 2 respectively. There is only one fixed point $E = Q = \infty$. The local coordinate is chosen to be Q. The normal bundles of the curves are then $\mathcal{O}(2)$ (since the order of poles of the normal bundle at $p = \infty$ cannot exceed the corresponding one of Q).

The solution to the Whitham hierarchy is recovered by expanding E and Q near $z = \infty$.

$$Q = p + \sum u_i p^{-1}, \quad E = \sum v_i Q^{-i} + x + Qy + Q^2 t. \tag{39}$$

By using $\Pi = \mathrm{d}E \wedge \mathrm{d}Q$ and the singular structure of Π, we see that the pull-back $\tilde{\Pi}$ has the form

$$\tilde{\Pi} = \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}\left(\frac{1}{2}p^2 + u_1\right) + \mathrm{d}t \wedge \mathrm{d}\left(\frac{1}{3}p^3 + pu_1 + w_1\right) \tag{40}$$

the simplicity of this 2-form then gives a solution to the dKP equation

$$((u_1)_t - u_1(u_1)_x)_x = (u_1)_{yy}. \tag{41}$$

Example 2. Elliptic curves.

We now look at the set of elliptic curves Γ_q in $\mathbb{CP}^1 \times \mathbb{CP}^1$, these are given the equation

$$w^2 = 4z^3 - g_1 z - g_2. \tag{42}$$

We take the twistor space to be $\mathbb{CP}^1 \times \mathbb{CP}^1$ with coordinates (w, z) and consider the deformations of the curve.

We first choose the 2-form to be $\mathrm{d}w \wedge \mathrm{d}z$, and we will choose the local coordinate k to be $\frac{w}{z}$. The marked point will be the point where $\mathrm{d}w \wedge \mathrm{d}z$ be comes singular on the curve, that is, the point (∞, ∞). We shall simply denote this point by ∞. Let λ be the uniformization parameter of the curve, and $\wp(\lambda)$ be the Weierstrass function. We choose our basis of cycle such that

$$\oint_a \mathrm{d}\lambda = \frac{2g_3^{-\frac{1}{10}} g_7^{-\frac{1}{15}}}{4}. \tag{43}$$

Since $z = \wp(\lambda)$ and $w = \wp'(\lambda)$, we see that the divisor D is of order 7 on the curve. The space $H^0(\Gamma_q, N_q)$ is therefore of dimension 7 (since there is no meromorphic 1-form with a simple pole on Γ_q, the dimension is 7 instead of 8). We also want to fix the marked point (∞, ∞) so we want all the curves to intersect this point. This means we only consider a six-dimensional subspace of $H^0(\Gamma_q, N_q)$. In particular, since the normal vector N such that μ_N has a pole of order 7 does not vanish at (∞, ∞), it will move the marked point and we shall not consider flows generated by such normal vectors.

We shall see in a moment that the full family of curves obtained from deforming (42) is given by

$$g_3 w^2 + g_4 w + g_5 wz + = g_6 z^3 + g_7 z^2 - g_1 z - g_2 \tag{44}$$

subject to a scale invariant such that

$$\sum g_i \partial_{g_i} = 0. \tag{45}$$

Each curve in (44) can be brought to the standard form (42) by a linear transformation

$$\begin{aligned} &w' = aw + bz + c, \quad z' = ez + f \\ &(w')^2 = 4(z')^3 - \tilde{g}_1 z' - \tilde{g}_2 \end{aligned} \tag{46}$$

therefore we can write the Weierstrass functions as

$$\wp(\lambda) = ez + f, \quad \wp'(\lambda) = aw + bz + c. \tag{47}$$

By comparing (46) with (44), we see that the coefficients $a, \ldots, f$ are given by

$$\begin{aligned} &a = g_3^{\frac{1}{2}}, \quad b = \frac{1}{2} g_5 g_3^{-\frac{1}{2}}, \quad c = \frac{1}{2} g_4 g_3^{-\frac{1}{2}}, \\ &e = \left(\frac{g_7}{4}\right)^{-\frac{1}{3}}, \quad f = \frac{1}{12\mathrm{e}^2}\left(g_8 + \frac{1}{4} g_5^2 g_3^{-1}\right) \\ &\tilde{g}_1 = \frac{1}{e}\left(g_1 + 12ef^2 - \frac{1}{4} g_4 g_5 g_3^{-1}\right) \\ &\tilde{g}_2 = g_2 - \tilde{g}_1 f + 4f^3 - c^2. \end{aligned} \tag{48}$$

We can therefore compute various powers of the jet k in terms of the Weierstrass function and hence compute the 1-forms $\mathrm{d}\Omega_{\infty,i}$. We shall now denote $\mathrm{d}\Omega_{\infty,i}$ by $\mathrm{d}\Omega_i$.

$$\begin{aligned} &\mathrm{d}\Omega_1 = \frac{2e}{a}(\wp(\lambda) - \omega)\mathrm{d}\lambda \\ &\mathrm{d}\Omega_2 = 4\left(\frac{e}{a}\right)^2\left(\wp'(\lambda) - \frac{b}{e}(\wp(\lambda) - \omega)\right)\mathrm{d}\lambda \\ &\mathrm{d}\Omega_3 = \left(\frac{e}{a}\right)^3\left[4\wp''(\lambda) - \frac{12b}{e}\wp'(\lambda) + \left(24f + 6\left(\frac{b}{e}\right)^2\right)(\wp(\lambda) - \omega)\right]\mathrm{d}\lambda \end{aligned}$$

$$
\begin{aligned}
\mathrm{d}\Omega_4 = \left(\frac{e}{a}\right)^4 \Bigg[& 32\wp'(\lambda)\wp(\lambda) - \frac{16b}{e}\wp''(\lambda) \\
& + \left(24\left(\frac{b}{e}\right)^2 + 64f\right)\wp'(\lambda) \\
& + \left(-96\frac{bf}{e} - 32c - 8\left(\frac{b}{e}\right)^3\right)(\wp(\lambda) - \omega)\Bigg]
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
\mathrm{d}\Omega_5 = \left(\frac{e}{a}\right)^5 \Bigg[& 16(\wp'(\lambda))^2 + 16\wp(\lambda)\wp''(\lambda) - 160\frac{b}{e}\wp'(\lambda)\wp(\lambda) \\
& + \left(40\left(\frac{b}{e}\right)^2 + 80f\right)\wp''(\lambda) + \left(-320\frac{bf}{e} - 80c - 40\left(\frac{b}{e}\right)^3\right)\wp'(\lambda) \\
& + \left(240\left(\frac{b}{e}\right)^2 f + 160\frac{bc}{e} - 16\tilde{g}_1 + 10\left(\frac{b}{e}\right)^4 + 480f^2\right)(\wp(\lambda) - \omega)\Bigg]
\end{aligned} \tag{50}
$$

where ω is the a-period of $\wp(\lambda)\mathrm{d}\lambda$ divided by the factor $\frac{2g_3^{-\frac{1}{10}}g_7^{-\frac{1}{15}}}{4}$ in (43). The holomorphic 1-form $\mathrm{d}\Omega_{h,1}$, which we will denote by $\mathrm{d}\Omega_0$ is

$$
\mathrm{d}\Omega_0 = \left(\frac{2g_3^{-\frac{1}{10}}g_7^{-\frac{1}{15}}}{4}\right)^{-1}\mathrm{d}\lambda. \tag{51}
$$

The Universal Whitham hierarchy that we are trying to solve is of the form

$$
\partial_j \mathrm{d}\Omega_i - \partial_i \mathrm{d}\Omega_j + \{\mathrm{d}\Omega_i, \mathrm{d}\Omega_j\} = 0. \tag{52}
$$

The coefficients of the expansion of the above equations can be expressed in terms of the g_i in (44). We will therefore treat the coefficients g_i as a solution to (52) and express them in terms of t_i. In fact, we will express t_i in terms of the g_i.

We first look at the tangent bundle of the curve Γ_q. By differentiating (44), we see that the tangent bundle is spanned by

$$
\Im = (2g_3w + g_4 + g_5z)\partial_z - (3z^2g_6 + 2zg_7 - g_1 - g_5w)\partial_w \tag{53}
$$

and the normal bundle of the curve Γ_q is spanned by

$$
\left\{w\partial_w - z\partial_z, \partial_w, z\partial_w, \partial_z, \frac{\partial_w}{(2g_3w + g_4 + g_5z)}, \frac{z\partial_w}{(2g_3w + g_4 + g_5z)}\right\} \tag{54}
$$

since the contraction of these vector fields with $\mathrm{d}w \wedge \mathrm{d}z$ gives 1-forms of desired order.

We can also see this as follows. Since the poles of $\mathrm{d}w$, $\mathrm{d}z$ are of order 4 and 3 at ∞, ∂_w and ∂_z vanish to the order 4 and 3 at ∞ respectively. Therefore the vector fields $w\partial_w$ etc. are holomorphic on Γ_q. However, the vector fields $w^2\partial_w$ and $z^2\partial_z$ which are holomorphic on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are not holomorphic on Γ_q. On the other hand, vector fields like $z\partial_w$ that are not holomorphic on $\mathbb{CP}^1 \times \mathbb{CP}^1$ become holomorphic on Γ_q as Γ_q only intersects $z = \infty$ and $w = \infty$ at (∞, ∞). Also, the vector fields $\frac{\partial_w}{(2g_3w+g_4+g_5z)}$ and $\frac{z\partial_w}{(2g_3w+g_4+g_5z)}$ are holomorphic as

∂_w is tangent to Γ_q when the denomination vanishes on Γ_q, which we can see by setting $\Im = 0$ in (53). Therefore the above are holomorphic normal bundles on Γ_q.

Since ∂_w acts on $\mathbb{CP}^1 \times \mathbb{CP}^1$ by an infinitesmal translation of w, we see that the normal vectors (54) generate the following movements of the curve.

$$
\begin{aligned}
&w\partial_w - z\partial_z = 5g_3\partial_{g_3} + 2g_4\partial_{g_4} + 3g_2\partial_{g_2} + 4g_1\partial_{g_1} + g_7\partial_{g_7} + 3g_5\partial_{g_5},\\
&\partial_w = -2g_3\partial_{g_4} + g_4\partial_{g_2} + g_5\partial_{g_1},\\
&z\partial_w = 2g_3\partial_{g_5} + g_4\partial_{g_1} - g_5\partial_{g_7},\\
&\partial_z = g_5\partial_{g_4} + 3g_6\partial_{g_7} - 2g_7\partial_{g_1} + g_1\partial_{g_2},\\
&\frac{\partial_w}{(2g_3w + g_4 + g_5z)} = \partial_{g_2},\\
&\frac{z\partial_w}{(2g_3w + g_4 + g_5z)} = \partial_{g_1}
\end{aligned}
$$

where we have used the scale invariance $\sum_i g_i\partial_{g_i} = 0$ to eliminate the $g_6\partial_{g_6}$ term in the first equation. We see that the full family of curves is of the form (44). By contracting the normal vectors with respect to the 2-form Π and using $\mu_{t_i} = -\mathrm{d}\Omega_i$, we see that

$$
\begin{aligned}
\partial_{g_1} &= \frac{1}{g_6}\partial_{t_1} + \partial_{g_1}t_0\partial_{t_0}\\
\partial_{g_2} &= g_3^{-\frac{3}{5}}g_6^{-\frac{2}{5}}\partial_{t_0}\\
\partial_{g_3} &= \frac{g_3}{5g_6}\partial_{t_5} + \frac{1}{4}\frac{g_5}{g_6^2}\partial_{t_4} - \frac{g_8}{3g_6^2}\partial_{t_3} + \partial_{g_3}t_0\partial_{t_0}\\
\partial_{g_4} &= \frac{1}{2g_6}\partial_{t_2} + \partial_{g_4}t_0\partial_{t_0}\\
\partial_{g_5} &= \frac{g_3}{4g_6^2}\partial_{t_4} + \frac{g_5}{3g_6^2}\partial_{t_3} - \frac{g_7}{2g_6^2}\partial_{t_2} + \partial_{g_5}t_0\partial_{t_0}\\
\partial_{g_7} &= -\frac{g_3}{3g_6^2}\partial_{t_3} - \frac{g_5}{2g_6^2}\partial_{t_2} + \frac{g_7}{g_6^2}\partial_{t_1} + \partial_{g_5}t_0\partial_{t_0}
\end{aligned}
\tag{55}
$$

while the functions $\partial_{g_i}t_0$ are expressed in terms of g_i and ω. From (55), we see that

$$
\begin{aligned}
t_1 &= \frac{2g_1 + g_7^2}{2g_6} + c_1\\
t_2 &= \frac{g_4 - g_5g_7}{2g_6} + c_2\\
t_3 &= -\frac{g_8g_3 + g_5^2}{3g_6^2} + c_3\\
t_4 &= \frac{g_3g_5}{4g_6^2} + c_4\\
t_5 &= \frac{g_3^2}{5g_6} + c_5
\end{aligned}
\tag{56}
$$

where c_i are integration constants. While t_0 satisfies the following equations

$$
\begin{aligned}
\partial_{g_1} t_0 &= g_3^{-\frac{3}{5}} g_6^{-\frac{2}{5}} \left[\omega e^{-1} - \frac{1}{3}\left(\frac{g_7}{g_6} + \frac{g_5^2}{4g_6 g_3}\right)\right] \\
\partial_{g_2} t_0 &= g_3^{-\frac{3}{5}} g_6^{-\frac{2}{5}} \\
\partial_{g_3} t_0 &= \frac{1}{5g_3}\left[\left(\frac{g_4^2}{g_3} - 3g_2 - \frac{g_7 g_5 g_4}{6g_6 g_3} + \frac{g_7 g_1}{3g_6} - \frac{g_5^3 g_4}{4g_3^2 g_6} + \frac{g_5^2 g_1}{2g_3 g_6}\right)\partial_{g_2} t_0 \right. \\
&\quad \left. + \left(\frac{g_4 g_5}{g_3} - 4g_1 - \frac{g_7 g_5^2}{6g_6 g_3} - \frac{2g_7^2}{3g_6} - \frac{g_5^4}{4g_3^2 g_6} - \frac{g_5^2 g_7}{g_6 g_3} + \frac{3g_5 g_4}{2g_3}\right)\partial_{g_1} t_0\right] \qquad (57) \\
\partial_{g_4} t_0 &= \frac{1}{2g_3}(-g_4 \partial_{g_1} t_0 - g_5 \partial_{g_2} t_0) \\
\partial_{g_5} t_0 &= \frac{1}{2g_3}\left[\left(\frac{g_5^2 g_4}{6g_6 g_3} - \frac{g_5 g_1}{3g_6}\right)\partial_{g_2} t_0 + \left(\frac{g_3^2}{6g_6 g_3} + \frac{2g_5 g_7}{3g_6} - g_4\right)\partial_{g_1} t_0\right] \\
\partial_{g_7} t_0 &= \frac{1}{3g_6}\left[\left(\frac{g_5 g_4}{2g_3} - g_1\right)\partial_{g_2} t_0 + \left(\frac{g_5^2}{2g_3} + 2g_7\right)\partial_{g_1} t_0\right]
\end{aligned}
$$

where $e = (4^{-1} g_6)^{\frac{1}{3}}$.

This gives a solution to the Universal Whitham hierarchy (52) given by twistor theory.

Example 3. Algebraic curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

This example generalized the last example to the case where the curves are given by a polynomial

$$P(w, z) = 0.$$

In this case we choose the symplectic form to be $\mathrm{d}z \wedge \mathrm{d}w$. The twistor space is then a subset of the normal bundle which consists of normal vectors T such that $\mu_T|_\Sigma$ are normalized 1-forms. This gives us the deformations of the curve Σ that correspond to the Whitham deformations.

Spectral curves of the dual isomonodromic deformations

The isomonodromic problem was studied by Jimbo et al. [16,17]. Suppose we have a linear system of ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi(z) = A(z)\Psi(z) \tag{58}$$

where $A(z)$ is an $n \times n$ matrix-valued rational function in z

$$A(z) = \sum_{\alpha \in \mathcal{D}} \frac{A_\alpha(z)}{(z-\alpha)^{r_\alpha}} + A_\infty(z)$$

where $A_\alpha(z)$ are polynomials in $z-\alpha$ of degrees $r_\alpha - 1$, $A_\infty(z)$ is a polynomial of degree r_∞ and $\mathcal{D}$ is a divisor in $\mathbb{CP}^1$. We call the system non-resonant if the leading terms of A_α have distinct eigenvalues when $r_\alpha > 1$ and that no eigenvalues of the leading terms of A_α differ by an integer when $r_\alpha = 1$. In this case, we can find unique formal solutions to the system of the form

$$\Psi(z-\alpha) = Y^{(\alpha)}(z)\exp(D_\alpha(z) + m_\alpha \log(z-\alpha)) \tag{59}$$

as $z \to \alpha$, where $Y^{(\alpha)}(z) = \sum_{j=0}^{\infty} Y_j^{(\alpha)}(z-\alpha)^{-j}$ is a formal Taylor series of matrix that is invertible at $z = \alpha$ and $D_\alpha(z)$ is a diagonal matrix with polynomial entries in $z-\alpha$. The matrix m_α is a diagonal matrix and is called the exponent of formal monodromy.

It is known (e.g. [31]) that near each pole α of $A(z)$, there exist sectors $S_\alpha^{(j)}$ on which solutions $\Psi_\alpha^{(j)}$ that are asymptotic to the one in (59) exist.

By comparing solutions on different sectors, one obtains the Stokes matrices $S_{jk}^{(\alpha)} = (\Psi_\alpha^{(j)})^{-1}\Psi_\alpha^{(k)}$. Similarly, analytic continuations of solutions near different poles define the connection matrices [18]. Let us denote the coefficients of D_α and the pole positions α by T collectively. The isomonodromic problem is to find the dependence of the matrix $A(z)$ on T such that the Stokes matrices, the connection matrices and the exponent of formal monodromy remain constant. It was shown in [16] that the dependence $A(z,T)$ of $A(z)$ on T is determined by the following differential equation of $A(z,T)$

$$\mathrm{d}A(z,T) = \partial_z B(z,T) - [A(z,T), B(z,T)] \tag{60}$$

where d is the exterior derivative with respect to T and $B(z,T)$ is a rational matrix 1-form.

The 'spectral curve' of the isomonodromic problem is then defined by the algebraic equation [15,27]

$$\det(w - A(z,T)) = 0. \tag{61}$$

Unlike the spectral curves of other integrable systems, the 'spectral curve' of an isomonodromic problem is not preserved by the isomonodromic flows. One can see this from (60). The term $[A(z,T), B(z,T)]$ on the right hand side of (60) would preserve the spectral invariance whereas the term $\partial_z B(z,T)$ would lead to deformations of the spectral curve.

From now on we will consider a simple case in which the matrix $A(z)$ is of the following form

$$A(z) = \sum_{i=1}^{r} \frac{A_i}{(z-\alpha_i)} + C \tag{62}$$

where A_i are rank 1 matrices constant in z and C is a diagonal matrix constant in z, $C = \mathrm{diag}(c_1,\ldots,c_n)$. The deformation parameters in this case are the α_i and the c_i.

In this case, a technique called 'duality' [15] can be employed to obtain an algebraic curve that is birational to the spectral curve. Moreover, this curve is defined by the zero locus of a polynomial $P(z,w) = 0$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$. These curves were considered by Sanguinetti and Woodhouse [27]. The explicit construction goes as follows. Let Δ be the diagonal matrix $\Delta = \mathrm{diag}(\alpha_1,\ldots,\alpha_r)$. Since all the A_i are of rank 1, we can write the matrix $A(z)$ as follows

$$A(z) = G^{\mathrm{T}}(\Delta - z)^{-1}F + C \tag{63}$$

where G, F are $n \times r$ matrices constant in z.

There is a 'dual isomonodromic problem' associated with it, which is defined by

$$\partial_w - F(c-w)^{-1}G^{\mathrm{T}} - \Delta. \tag{64}$$

It was shown [15] that when $A(z)$ is deformed isomonodromically, the dual system also undergoes isomonodromic deformation.

The *dual isomonodromic spectral curve* is defined by the following polynomial in w and z

$$\det\left[\mathbb{M}-\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}\right]=0 \tag{65}$$

where $\mathbb{M}$ is the following matrix

$$\mathbb{M}=\begin{pmatrix} \Delta & -F \\ G^t & C \end{pmatrix}.$$

The genus of such a curve is shown in [27] to be $(n-1)(r-1)$. The whole set of curves Σ_T obtained by deforming the spectral curve through isomonodromic deformations forms an $n+r$ parameter family of curves. This is a subset of curves defined by the zero locus of all polynomials

$$\sum_{i=0}^{r}\sum_{j=0}^{n} X_{ij} z^i w^j = 0$$

which is an $nr = 2n + 2r + g$ parameter family of curves. We can treat this family of curves and the two-dimensional complex manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a twistor space of the Universal Whitham hierarchy as in Example 3. The 2-form is given by $\mathrm{d}z \wedge \mathrm{d}w$ and the fixed points are the intersection points between $z = \infty$, $w = \infty$ and the curves. These are the points $(z = \infty, w = c_i)$ and $(z = \alpha_i, w = \infty)$. The local coordinates near $z = \infty$ are chosen to be z^{-1} and the local coordinates near $w = \infty$ are chosen to be w^{-1}. In this case there is no natural choice of the point P_1 and we will choose it to be $(z = \alpha_1, w = \infty)$.

In this case the Universal Whitham times can be solved by the hodograph method [23,30]. They are given by

$$\begin{aligned} t_{\alpha_i,j} &= \mathrm{res}_{w=\infty} w^{-j} z\mathrm{d}w, \quad j = 0, 1 \\ t_{c_i,j} &= -\mathrm{res}_{z=\infty} z^{-j} w\mathrm{d}z, \quad j = 0, 1 \\ t_{h,j} &= \oint_{a_j} z\mathrm{d}w, \quad j = 1, \ldots, g \end{aligned} \tag{66}$$

where we used the index α_i to denote the point $(z = \alpha_i, w = \infty)$ and the index c_i to denote the point $(z = \infty, w = c_i)$. The residues are taken around the corresponding points. The integrals defining the $t_{h,j}$ are around a certain choice of a-cycles on the curve. Note that these formula are independent of the choice of P_1. The flows along the $t_{\alpha_i,j}$ and $t_{c_i,j}$ generate the Whitham flows that comes from the Whitham averaging process, together with the extra flows along $t_{h,j}$, they form the Universal Whitham hierarchy which generates all the deformations of the curve.

Since the Universal Whitham flows generate all the deformations of the algebraic curve, we can express the isomonodromic deformations of the spectral curve in terms of the Universal Whitham flows.

$$\begin{aligned} \frac{\partial}{\partial \alpha_i} &= \sum_A \frac{\partial t_A}{\partial \alpha_i}\partial_A \\ \frac{\partial}{\partial c_i} &= \sum_A \frac{\partial t_A}{\partial c_i}\partial_A. \end{aligned} \tag{67}$$

To see how the isomonodromic times are related to the Universal Whitham times, we first consider the behavior of w near $(z = \infty, w = c_i)$. Near these points, w has the Laurent expansion

$$w = c_i + m^i_\infty z^{-1} + O(z^{-2}) \tag{68}$$

where m^i_∞ is the ith entry in the formal exponent of monodromy at $z = \infty$. From (66) we see that

$$t_{c_i,0} = -m^i_\infty, \quad t_{c_i,1} = -c_i. \tag{69}$$

Similarly, near $(z = \alpha_i, w = \infty)$, z has the expansion

$$z = \alpha_i + n^i_\infty w^{-1} + O(w^{-2}) \tag{70}$$

where n^i_∞ is the ith entry in the formal exponent of monodromy at $w = \infty$ of the dual system (64). We see that

$$t_{\alpha_i,0} = n^i_\infty, \quad t_{\alpha_i,1} = \alpha_i. \tag{71}$$

By substituting these into (67) we obtain

$$\begin{aligned}
\frac{\partial}{\partial \alpha_i} &= \partial_{\alpha_i,1} + \sum_{k=1}^{g} \frac{\partial}{\partial \alpha_i}\left(\oint_{a_k} z\mathrm{d}w\right)\partial_{h,k} \\
\frac{\partial}{\partial c_i} &= \partial_{c_i,1} + \sum_{k=1}^{g} \frac{\partial}{\partial c_i}\left(\oint_{a_k} z\mathrm{d}w\right)\partial_{h,k}.
\end{aligned} \tag{72}$$

To calculate the second terms, first note that since z is single value on the spectral curve, the derivatives $\frac{\partial}{\partial c_i}(\oint_{a_k} z\mathrm{d}w)$ and $\frac{\partial}{\partial \alpha_i}(\oint_{a_k} z\mathrm{d}w)$ has no boundary terms. That is, we have

$$\frac{\partial}{\partial c_i}\left(\oint_{a_k} z\mathrm{d}w\right) = \oint_{a_k} \frac{\partial}{\partial c_i} z\mathrm{d}w \tag{73}$$

$$\frac{\partial}{\partial \alpha_i}\left(\oint_{a_k} z\mathrm{d}w\right) = \oint_{a_k} \frac{\partial}{\partial \alpha_i} z\mathrm{d}w. \tag{74}$$

To evaluate $\partial_{c_i} z\mathrm{d}w$ or $\partial_{\alpha_i} z\mathrm{d}w$, we can either think of the isomonodromic flows as flows that keep z fixed and change w as in the original isomonodromic system (58), or we could think of these flows as keeping w fix and changing z as in the dual isomonodromic system (64). To see that these two different interpretations make no difference in the final result, let us denote the determinant in (65) by $P(z, w, T)$. Let

$$P(z, w, T) = \sum_{i=0}^{r}\sum_{j=0}^{n} X_{ij}(T) z^i w^j = 0.$$

Then we see that

$$\partial_{\alpha_i} z = -\left(\sum_{i=0}^{r}\sum_{j=0}^{n} \partial_{\alpha_i} X_{ij}(T) z^i w^j\right)(\partial_z P(z, w, T))^{-1}$$

$$\partial_{\alpha_i} w = -\left(\sum_{i=0}^{r}\sum_{j=0}^{n}\partial_{\alpha_i}X_{ij}(T)z^i w^j\right)(\partial_w P(z,w,T))^{-1}$$

and similar expressions for $\partial_{c_i} w$ and $\partial_{c_i} z$. If we think of the isomonodromic flows as flows that fix w, the integrals in (73) are then

$$\oint_{a_k}\frac{\partial}{\partial\alpha_i}z\mathrm{d}w = -\oint_{a_k}\left(\sum_{i=0}^{r}\sum_{j=0}^{n}\partial_{\alpha_i}X_{ij}(T)z^i w^j\right)(\partial_z P(z,w,T))^{-1}\mathrm{d}w \tag{75}$$

on the other hand, if we think of the flows as flows that fix z, the integrals become

$$\oint_{a_k}\frac{\partial}{\partial\alpha_i}z\mathrm{d}w = \oint_{a_k}\left(\sum_{i=0}^{r}\sum_{j=0}^{n}\partial_{\alpha_i}X_{ij}(T)z^i w^j\right)(\partial_w P(z,w,T))^{-1}\mathrm{d}z \tag{76}$$

since $\oint_{a_k} z\mathrm{d}w = -\oint_{a_k} w\mathrm{d}z$ as $z\mathrm{d}w + w\mathrm{d}z$ is a total differential.

To see that the right hand sides of (75) and (76) are the same, note that by differentiating the expression $P(z,w,T) = 0$, we have $(\partial_w P(z,w,T))^{-1}\mathrm{d}z = -(\partial_z P(z,w,T))^{-1}\mathrm{d}w$. By replacing the α_i derivatives in (75) and (76) by derivatives of c_i and applying a similar argument, we see that the same holds for the c_i derivatives.

To compute the derivatives of the X_{ij}, recall that an isomonodromic deformation changes the matrix $A(z)$ in the following way

$$\mathrm{d}A(z,T) = \partial_z B(z,T) - [A(z,T), B(z,T)].$$

The matrix 1-form $B(z,T)$ in this case is

$$B(z,T) = \sum_{i=1}^{r} B_{\alpha_i}(z,T)\mathrm{d}\alpha_i + \sum_{i=1}^{n} B_{c_i}(z,T)\mathrm{d}c_i$$

where the $B_{\alpha_i}(z,T)$ and the $B_{c_i}(z,T)$ are as follows [15]

$$B_{\alpha_i}(z,T) = -\frac{A_{\alpha_i}}{(z-\alpha_i)} \tag{77}$$

$$B_{c_i}(z,T) = zE_i + \sum_{j\neq i}\sum_{k=1}^{r}\frac{E_i A_{\alpha_k}E_j + E_j A_{\alpha_k}E_i}{\alpha_i - \alpha_j} \tag{78}$$

where E_i are the $n\times n$ diagonal matrices with 1 on the ith entry and zero elsewhere.

The key observation here is that $\partial_z B_{\alpha_i} = \partial^e_{\alpha_i}A(z,T)$ and $\partial_z B_{c_i} = \partial^e_{c_i}A(z,T)$, where $\partial^e_{\alpha_i}$ and $\partial^e_{c_i}$ denotes the explicit derivatives with respect to α_i and c_i.

Therefore we have

$$\partial_i A(z,T) = \partial^e_i A(z,T) - [A(z,T), B_i(z,T)]$$

where the index i will be used to denote α_i or c_i.

Since the commutator term $[A(z,T), B_i(z,T)]$ fixes the spectral invariance of the matrix $A(z,T)$ and it is known that the coefficients $X_{ij}(T)$ of the spectral curve are spectrally invariant, [15] we see that the derivatives $\partial_T X_{ij}$ in (76) or (75) are the same as their explicit derivatives with respect to the times T. That is

$$\partial_T X_{ij} = \partial^e_T X_{ij} \tag{79}$$

this observation simplifies the calculations significantly as the explicit dependences of the coefficients X_{ij} on α_i and c_i are polynomials, which can be seen from (65).

In summary, the isomonodromic flows and the Universal Whitham flows are related by

Proposition 8. *Let $A(z)$ be a matrix of the form*

$$A(z) = G^{\mathrm{T}}(\Delta - z)^{-1}F + C$$

where G, F are $n \times r$ matrices constant in z, $\Delta = \mathrm{diag}(\alpha_1, \ldots, \alpha_r)$ and $C = \mathrm{diag}(c_1, \ldots, c_n)$.

Define the spectral curve of $A(z)$ by

$$\det\left[\mathbb{M} - \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}\right] = 0$$

where $\mathbb{M}$ is the following matrix

$$\mathbb{M} = \begin{pmatrix} \Delta & -F \\ G^t & C \end{pmatrix}$$

then, when the system of linear ODE

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = A(z, T)Y$$

deforms isomonodromically with deformation parameters α_i and c_i, the deformations of the spectral curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be expressed in terms of the Universal Whitham flows (defined in Definition 3) as follows

$$\partial_{\alpha_i} = \partial_{\alpha_i,1} + \sum_{k=1}^{g} \oint_{a_k} \left(\sum_{i=0}^{r} \sum_{j=0}^{n} \partial^e_{\alpha_i} X_{ij}(T) z^i w^j \right) P_w^{-1} \mathrm{d}z \partial_{h,k}$$

$$\partial_{c_i} = \partial_{c_i,1} + \sum_{k=1}^{g} \oint_{a_k} \left(\sum_{i=0}^{r} \sum_{j=0}^{n} \partial^e_{c_i} X_{ij}(T) z^i w^j \right) P_w^{-1} \mathrm{d}z \partial_{h,k} \tag{80}$$

where ∂^e_i denotes the explicit derivative and P_w is the w derivative of $P(z, w, T)$. The flows $\partial_{\alpha_i,1}$ and $\partial_{c_i,1}$ are the Whitham flows that come from the Whitham averaging. The isomonodromic flows coincide with the Whitham flows if and only if the summation parts in (80) vanish.

Here the twistor space of the Universal Whitham hierarchy is taken to be the set of algebraic curves given by the zero locus of polynomials $P(z, w)$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$, where $P(z, w)$ has degree at most n in w and r in z. The marked points are the points where z or w is infinity.

Remark. As one could see from (80), the contraction $\partial_i \lrcorner \Pi$ of ∂_i with the 2-form $\mathrm{d}z \wedge \mathrm{d}w$ would result in a 1-form on the curve that is holomorphic everywhere apart from a second order pole at the point $(z = \alpha_i, w = \infty)$ for ∂_{α_i} or $(z = \infty, w = c_i)$ for ∂_{c_i}. In [28,29], Takasaki raised the question of whether it is possible to choose a canonical basis of cycles that could make these 1-forms normalized. This is the same as choosing a basis of cycles so that the summation parts in (80) vanish. This example does not imply that the isomonodromic flows and the Whitham flows coincide. However, by the observation (79), we show that the formula in [28,29] can be greatly simplified.

We will now give an example.

Example 4. Painlevé V.

The Painlevé V equation is the following second order nonlinear ODE

$$u'' = (u')^2\left(\frac{1}{2u} + \frac{1}{u-1}\right) - \frac{u'}{t} + \frac{(u-1)^2}{t^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{t} + \frac{\delta u(u+1)}{u-1}$$

where α, β, γ and δ are constants. It can be represented as the isomonodromic deformations of a system of linear ODEs in which $A(z)$ is of the following form [18]

$$A(z) = \frac{A_1}{z-1} + \frac{A_0}{z} + \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}.$$

We will follow the approach in [15] to construct the spectral curve. We first set the constant δ to be $\delta = 2$. In this case, the matrices A_0 and A_1 will be of rank 1 and a factorization in (63) can be carried out.

In [15], the Painlevé V was represented by the isomonodromic deformations of the system of linear ODEs

$$\begin{aligned}\frac{\mathrm{d}Y}{\mathrm{d}z}Y^{-1} = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} &+ 2z^{-1}\begin{pmatrix} -x_1y_1 - \mu_1 & -y_1^2 + \dfrac{\mu_1^2}{x_1^2} \\ x_1^2 & x_1y_1 - \mu_1 \end{pmatrix} \\ &+ 2(z-1)^{-1}\begin{pmatrix} -x_2y_2 - \mu_2 & -y_2^2 + \dfrac{\mu_2^2}{x_2^2} \\ x_2^2 & x_2y_2 - \mu_2 \end{pmatrix}\end{aligned}$$

in which μ_1, μ_2 and $c = \frac{1}{2}(x_1y_1 + x_2y_2)$ are constants. These constants are related to α, β and γ as follows

$$\alpha = \frac{\mu_2^2}{2}, \quad \beta = -\frac{\mu_1^2}{2}, \quad \gamma = 4c + 2.$$

The solution y to the Painlevé V equation is given by $u = -\frac{x_1^2}{x_2^2}$.

The factorization (63) is given by

$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} x_1 & y_1 - \dfrac{\mu_1}{x_1} \\ x_2 & y_2 - \dfrac{\mu_2}{x_2} \end{pmatrix}, \quad G = \frac{1}{\sqrt{2}}\begin{pmatrix} y_1 + \dfrac{\mu_1}{x_1} & -x_1 \\ y_2 + \dfrac{\mu_2}{x_2} & -x_2 \end{pmatrix}$$

with $\Delta = \mathrm{diag}(0, 1)$.

The spectral curve is then

$$\begin{aligned}P(z, w) = z^2w^2 - w^2z + (\mu_1)wz - (\mu_1 + \mu_2)w - t^2z^2 + (t^2 + tx_1y_1)z \\ - 2ct + \frac{1}{4}\left((x_1y_2 - x_2y_1)^2 + \left(\frac{\mu_1^2}{u} + \mu_2^2u\right)\right) = 0.\end{aligned}$$

This curve is of genus 1. The fixed points are $(z = 0, w = \infty)$, $(z = 1, w = \infty)$, $(z = \infty, w = t)$ and $(z = \infty, w = -t)$. We can now apply (80) to express ∂_t in terms of the Universal Whitham flows

$$\partial_t = \partial_{t,1} + \oint_a (-2tz^2 + (2t + x_1 y_1)z - 2c) P_w^{-1} \mathrm{d}z \partial_{h,1}$$

where a is an a-cycle chosen on the curve.

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